Kyle's Model

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Broad Lesson Plan

Introduction

- How does private information get impounded into the price of a security?
- Informed traders, uninformed traders, and market makers
- Strategic and rational behaviors in equilibrium
- Main reference

  Empirical Market Microstructure: The Institutions, Economics, and Econometrics of Securities Trading, Joel Hasbrouck, Oxford University Press (2007)

Setting

- The 1-period terminal security value

$$v \sim N\left(p_0, \Sigma_0\right)$$

- One informed trader who knows v and enters a demand x.
- Liquidity ("noise") traders submit a net order flow

$$u \sim N(0, \sigma_u^2),$$

  which is independent of v.
- The market maker (MM) observes the total demand $y = x + u$ and then sets a price, p.
- All of the trades are cleared at p.

Informed Trader's Profit

- The informed trader conjectures that the MM uses a linear price adjustment rule

$$p = \lambda y + \mu,$$

where y is the total order flow: $y = u + x$.

- λ is Kyle's inverse measure of liquidity.
- The informed trader's profits are $\pi = (v - p)x$.
- Substituting in for the price conjecture and y yields

$$\pi = x[v - \lambda(u + x) - \mu].$$

- The expected profits are $\mathbb{E}(\pi) = x(v - \lambda x - \mu)$.

Optimization by Informed Trader

- If the informed trader is buying ($x > 0$), it is possible that a large surge of uninformed buying ($u > 0$) drives the $\lambda(u + x) + \mu$ above v.
- So, the informed trader chooses x to maximize $\mathbb{E}(\pi)$, yielding

$$x = \frac{v - \mu}{2\lambda}.$$

- The second-order condition is $\lambda > 0$.
- The larger λ is, the larger will be the impact of order flow.

Market Maker's Strategy

- The MM conjectures that the informed trader's demand is linear in v:

$$x = \alpha + \beta v.$$

- Being mindful of the optimization process that the informed trader followed, the MM can solve for α and β for all v:

$$\frac{v - \mu}{2\lambda} = \alpha + \beta v.$$

- Hence,

$$\alpha = -\frac{\mu}{2\lambda} \quad \text{and} \quad \beta = \frac{1}{2\lambda}. \tag{1}$$

Bivariate Normal Projection

- Suppose that A and B are bivariate normal random variables with means μ_A and μ_B, variances σ_A^2 and σ_B^2, and covariance σ_{AB}.
- The conditional expectation of B given A is

$$\mathbb{E}\left[B \middle| A = a\right] = \mu_B + \frac{\sigma_{AB}}{\sigma_A^2}\left(a - \mu_A\right).$$

- Because this conditional expectation is linear in A, it is equivalent to projection.
- The variance of the projection error is

$$\mathbb{V}\left[B \middle| A = a\right] = \sigma_B^2 - \frac{\sigma_{AB}^2}{\sigma_A^2}.$$

Applying the Projection

- Here, given the definition of the order flow variable and the MM's conjecture about the informed traders behavior,

$$y = u + \alpha + \beta v,$$

we have

$$\mathbb{E}\left(y\right) = \alpha + \beta\,\mathbb{E}\left(v\right) = \alpha + \beta p_0, \quad \mathbb{V}(y) = \sigma_u^2 + \beta^2 \Sigma_0, \quad \text{and} \quad \mathbb{C}(y, v) = \beta \Sigma_0.$$

- Using these in the projection results gives

$$\mathbb{E}[v|y] = p_0 + \frac{\beta \Sigma_0 (y - \alpha - \beta p_0)}{\sigma_u^2 + \beta^2 \Sigma_0}$$

$$\mathbb{V}[v|y] = \frac{\sigma_u^2 \Sigma_0}{\sigma_u^2 + \beta^2 \Sigma_0} \qquad (2)$$

Solution

- This must equal $p = \lambda y + \mu$ for all values of y. Hence

$$\mu = \frac{-\alpha\beta\Sigma_0 + \sigma_u^2 p_0}{\sigma_u^2 + \beta^2\Sigma_0} \quad \text{and} \quad \lambda = \frac{\beta\Sigma_0}{\sigma_u^2 + \beta^2\Sigma_0} \tag{3}$$

- Solving equations (1) and (3) yields

$$\alpha = -p_0\sqrt{\frac{\sigma_u^2}{\Sigma_0}} \quad \text{and} \quad \beta = \sqrt{\frac{\sigma_u^2}{\Sigma_0}}$$

as well as

$$\mu = p_0 \quad \text{and} \quad \lambda = \frac{1}{2}\sqrt{\frac{\Sigma_0}{\sigma_u^2}}$$

Discussion

- The informed trader's expected profit is

$$\mathbb{E}(\pi) = \frac{(v - p_0)^2}{2} \sqrt{\frac{\sigma_u^2}{\Sigma_0}}.$$

- $\mathbb{E}(\pi)$ increases with the divergence between the true value $v - p_0$ and the unconditional mean σ_u^2.
- Noise trading as providing camouflage for the informed trader.
- How much of the private information is impounded in the price?

$$\mathbb{V}[v|p] = \mathbb{V}[v|y] = \frac{\Sigma_0}{2}.$$

Transient versus Permanent Price Impacts

- Bid ask bounce is **transient** and ephemeral. Due to private information, trades also have effects on prices that are cumulative and **permanent**.
- As a general trend, active buys (sells) can drive the price up (down).
- Suppose that a buy order at 10am drives the price up by \$0.05, and the closing price at 16:00 happens to be \$10.00. We conjecture that if that trade had not occurred, the closing price would have been \$9.95.
- How permanent is "permanent"? This is debatable. For present purposes, "permanent" means "does not go away over reasonable trading horizons."

Permanent Price Impact Model

- We can modify the random walk model to incorporate a permanent price impact. Let

$$
\begin{aligned}
S_t &:= \text{net order flow in shares} \\
&= \text{\# shares actively purchased} - \text{\# shares actively sold.}
\end{aligned}
$$

- Then, the update of log price is

$$p_t = p_{t-1} + \alpha + \lambda S_t + u_t,$$

where λ is Kyle's price impact parameter.

- How can this model be incorporated into a trading model?

Two-Period Case

- Problem: minimize the cost of buying S_{Total} shares over two periods, ("period 0 is "right now.") Consider the cost

$$C = p_1 S_1 + p_2 S_2.$$

- By construction, we have

$$\begin{aligned} p_1 &= \alpha + p_0 + \lambda S_1 + u_1 \\ p_2 &= \alpha + p_1 + \lambda S_2 + u_2 \\ &= 2\alpha + p_0 + \lambda S_1 + \lambda S_2 + u_1 + u_2 \end{aligned}$$

- Substitute into the cost expression to obtain

$$C = S_1\big(\alpha + p_0 + \lambda S_1 + u_1\big) + S_2\big(2\alpha + p_0 + \lambda S_1 + \lambda S_2 + u_1 + u_2\big).$$

- The expected cost is

$$\mathbb{E}(C) = S_1\big(\alpha + p_0 + \lambda S_1\big) + S_2\big(2\alpha + p_0 + \lambda S_1 + \lambda S_2\big).$$

Problem

- Minimize $\mathbb{E}(C)$ subject to $S_1 + S_2 = S_{\text{total}}$.
- Rewrite the constraint as $S_2 = S_{\text{total}} - S_1$ and substitute into the expected cost, we obtain

$$\mathbb{E}(C) = \lambda S_1^2 - S_1\big(\alpha + \lambda S_{\text{total}}\big) + S_{\text{total}}\big(2\alpha + p_0 + \lambda S_{\text{total}}\big).$$

- Now the expression involves just S_1. The first order condition is

$$\frac{d\mathbb{E}(C)}{dS_1} = -\alpha + 2\lambda S_1 - \lambda S_{\text{total}} = 0.$$

Solution

- The optimal number of shares to buy in period 1 is

$$S_1^\star = \frac{\alpha + \lambda S_{\text{total}}}{2\lambda}.$$

- Since $S_1 + S_2 = S_{\text{total}}$, the optimal number of shares to buy in period 2 is

$$S_2^\star = -\frac{\alpha - \lambda S_{\text{total}}}{2\lambda}.$$

- Therefore, the expected cost with optimal buying is

$$\mathbb{E}(C^\star) = -\frac{(\alpha + \lambda S_{\text{total}})^2}{4\lambda} + S_{\text{total}}\big(2\alpha + p_0 + \lambda S_{\text{total}}\big).$$

Variance of Optimal Cost

- The original formula for C, evaluated at the optimal quantities is

$$C^\star = \frac{1}{4} S_{\text{total}} \big(4p_0 + 3\lambda S_{\text{total}} + 4u_1 + 2u_2 \big)$$

- It follows that the variance of $C^\star$ is

$$\mathbb{V}\big(C^\star\big) = S_{\text{total}}^2 \left(\mathbb{V}(u_1) + \frac{1}{4} \mathbb{V}(u_2) \right) = \frac{5}{4} S_{\text{total}}^2 \sigma_u^2.$$

General Case

- The analysis extends to more than two periods. For example, for purchase over three periods,

$$S_1^\star = \frac{\alpha}{\lambda} + \frac{S_{\text{total}}}{3}, \qquad S_2^\star = \frac{S_{\text{total}}}{3}, \qquad S_3^\star = -\frac{\alpha}{\lambda} + \frac{S_{\text{total}}}{3}.$$

- If $\alpha = 0$, the optimal strategy over T periods is simply a constant when S_{total} and T are given. Namely,

$$S_t^\star = \frac{S_{\text{total}}}{T},$$

for $t = 1, 2, \ldots, T$.

Results for the General Case

- In generally, under the constant-size trading strategy and setting $\alpha = 0$, the price for each time step is

$$p_t = p_0 + \frac{\lambda S_{\text{total}}}{T} t + u_1 + u_2 + \cdots + u_t = p_0 + \frac{\lambda t S_{\text{total}}}{T} + \sum_{s=1}^{t} u_s$$

- Accordingly, the expected cost under the optimal buying strategy is

$$\mathbb{E}(C^\star) = p_0 S_{\text{total}} + \frac{(T+1)\lambda S_{\text{total}}^2}{2T}.$$

- Moreover, the variance is computed as

$$\mathbb{V}(C^\star) = \frac{S_{\text{total}}^2 \sigma_u^2}{T^2} \sum_{t=1}^{T} t^2 = \frac{S_{\text{total}}^2 \sigma_u^2}{T^2} \left(\frac{T(T+1)(2T+1)}{6} \right)$$
$$= \frac{(T+1)(2T+1)}{6T} S_{\text{total}}^2 \sigma_u^2.$$

Insight!

- As T goes up, expected cost goes down, but risk rises.
- In portfolio theory there is a fundamental trade-off between risk and return. There is a similar trade-off here. In portfolio theory the trade-off line is called the "efficient portfolio frontier."
- Here it is sometimes called the efficient trading frontier.

Estimation of λ

- The general relationship we are trying to estimate is

$$p_t = p_{t-1} + \alpha + \lambda S_t + u_t,$$

or

$$\Delta p_t = \alpha + \lambda S_t + u_t.$$

- This specification can be estimated with intraday, high-frequency datasets.

- These datasets, however, are large and complex. With some simplifying assumptions, we can form an estimate from the more common and tractable daily data.

Daily Estimated λ

- On average $\alpha \approx 0$ and $u_t \approx 0$. If we assume that these approximations hold period-by-period, we can write

$$\Delta p_t = \lambda S_t.$$

- Taking the absolute value of each side, we obtain $|\Delta p_t| = \lambda |S_t|$.
- Now S_t is the order flow volume over the interval. If the interval is short, we can approximate its absolute value by the trading volume, $V_t \approx |S_t|$.
- Rearranging, the approximate λ value is

$$\lambda = \frac{|\Delta p_t|}{V_t}.$$

- What is the intuitive interpretation of this approximate formula?

Amihud's Illiquidity Measure

- This result suggests estimating λ by the absolute change of log price and volume as follows:

$$\widehat{\lambda} = \overline{\left(\frac{|\Delta p_t|}{V_t}\right)}$$

- The overline indicates an average taken over a sample of intervals.
- Since price and volume data are widely reported at a daily frequency, it is convenient to use a sample of daily data to estimate λ.
- The $\widehat{\lambda}$ statistic suggested above is a variant of the Amihud illiquidity measure [Amihud, 2002].

Assumptions in Estimation Using Intraday Data

- Even when we estimate $\Delta p_t = \alpha + \lambda S_t + u_t$ with high-frequency data, we need to make several simplifications.

- The price jump caused by an incoming order does not always take place in the same millisecond, or second. So we use larger time intervals. Here, we will let t index minutes.

- The trade price itself contains a bid-ask bounce. The bounce tends to wash out over time, but not necessarily within the minute. So we use the bid-ask midpoint for p_t.

Intraday Econometric Model Using Order Flows

- Therefore, we estimate the following regression instead

$$\Delta m_t = \alpha + \lambda \text{NBV}_t + u_t,$$

where

  - t is minutes
  - m_t is the quote midpoint at the end of minute t;
  - NBV_t is the order flow over the minute

- Sometimes, to avoid giving too much weight to large trades, we let the driving variable be NB_t, the net number of buys in the minute (the sum of the trade signs). Hence,

$$\Delta m_t = \alpha + \lambda \text{NB}_t + u_t.$$

Takeaways

- A model of linear equilibrium between informed trader and market maker
- Kyle's λ is a measure of (inverse) liquidity, which can be estimated easily.
- Informed traders' expected profit increases with the variance of uninformed traders' order flow.
- Order flow drives the security's price—linear price impact of order flow
- Can the price impact function be nonlinear?

References

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